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Wishart and Chi-Square Distributions Associated with Matrix Quadratic Forms

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For a normally distributed random matrix Y with a general variance–covariance matrix Σ_Y, and for a nonnegative definite matrix Q, necessary and sufficient conditions are derived for the Wishartness of $Y'QY$. The conditions resemble those obtained by Wong, Masaro, and Wang (1991, *J. Multivariate Anal.* **39**, 154–174) and Wong and Wang (1993, *J. Multivariate Anal.* **44**, 146–159), but are verifiable and are obtained by elementary means. An explicit characterization is also obtained for the structure of Σ_Y under which the distribution of $Y'QY$ is Wishart. Assuming Σ_Y positive definite, a necessary and sufficient condition is derived for every univariate quadratic from $l'Y'QYl$ to be distributed as a multiple of a chi-square. For the case $Q = I_n$, the corresponding structure of Σ_Y is identified. An explicit counterexample is constructed showing that Wishartness of $Y'Y$ need not follow when, for every vector l, $l'Y'Yl$ is distributed as a multiple of a chi-square, complementing the well-known counterexample by Mitra (1969, *Sankhyā A* **31**, 19–22). Application of the results to multivariate components of variance models is briefly indicated.

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1. INTRODUCTION

Consider n observation vectors $y_1, ..., y_n$, each of dimension p, such that the $n \times p$ matrix

$$Y = (y_1, ..., y_n)'$$

follows a normal distribution with mean zero and variance-covariance matrices given by

$$\mathrm{Var}(y_i) = \Sigma_{ii}, \qquad \mathrm{Cov}(y_i, y_j) = \Sigma_{ij}. \tag{1.1}$$

The matrices Σ_{ii} and Σ_{ij} are thus $p \times p$ matrices ($i, j = 1, ..., n$). We shall denote by Σ_Y the $np \times np$ matrix whose (i, j)th block is Σ_{ij} and write

$$Y \sim N(0, \Sigma_Y),$$

where Σ_Y is actually the covariance matrix of the $np \times 1$ vector $\mathrm{vec}(Y') = (y'_1, ..., y'_n)'$.

Now let Q be a nonnegative definite $n \times n$ matrix. The purpose of this article is, first, to obtain necessary and sufficient conditions under which

(i) $Y'QY$ follows a Wishart distribution;

and, second, to obtain necessary and sufficient conditions under which

(ii) $l'Y'QYl$ is distributed as a multiple of a chi-square distribution for every vector l.

The former problem has been previously addressed in the literature, some recent references being Wong, Masaro, and Wang (1991) and Wong and Wang (1993, 1995); see also Pavur (1987) and Boik (1988). (For further references to the relevant literature, refer to these articles.) The results in Pavur (1987) and Boik (1988) assume that Σ_Y is positive definite. This condition is relaxed in Wong, Masaro and Wang (1991) and Wong and Wang (1993). However, their characterizations of the Wishartness of $Y'QY$ appear to involve unnecessarily tedious derivations, and, more importantly, are not easily verified. In particular, the proof of the characterizations contained in Proposition 3.1 in Wong and Wang (1993) relies on, among other things, analytic continuation of moment generating functions. Also, their characterization provides conditions under which $Y'QY \sim W_p(m, \Sigma)$ for *specified* m and Σ; i.e., the conditions are not verifiable. However, as pointed out in Wong and Wang (1993), once m and Σ are specified, their condition can be verified (see also Remark 1 in the next section). Here $W_p(m, \Sigma)$ is the Wishart distribution with m degrees of freedom and scale matrix Σ. Recall that $W_p(m, \Sigma)$ is the distribution of the random matrix $\sum_{i=1}^{m} z_i z_i'$ where $z_i \sim N(0, \Sigma)$ and the z_i's are independently distributed $(i = 1, ..., m)$.

Instead we shall derive conditions under which $Y'QY$ follows *some* Wishart distribution, without specifying m and Σ. These quantities can indeed be computed, once it is known that Wishartness holds. The characterizations of Wishartness of $Y'QY$, given in Section 2 (Theorems 1 and 2, Corollary 1), are all derived using only elementary properties of the Wishart distribution. Corollary 1 provides an explicit and complete characterization of the class of permissible structures of Σ_Y under which the distribution of $Y'QY$ is Wishart.

In Section 2 we also address the latter problem and derive, under the assumption that Σ_Y is positive definite, a necessary and sufficient condition for property (ii) to hold (Theorem 3). The problem of extending this result to nonnegative definite Σ_Y appears complicated and remains unsolved. Specializing to the case $Q = I_n$, we also obtain a complete characterization of the structures of Σ_Y under which every quadratic form of $Y'Y$ is distributed as a multiple of a chi-square (Corollary 2). As a bonus, this characterization suggests a simple counterexample showing that, for a random $p \times p$ matrix S, Wishartness of S does not follow from the fact that every quadratic form of S is distributed as a multiple of a chi-square (Example 1). This counterexample complements the one given by Mitra (1969), in that it is by explicit construction and, furthermore, shows that the result fails even when the random matrix S is of the form $S = Y'Y$, with Y a normally distributed random matrix. A sufficient condition on the structure of Σ_Y is given under which Wishartness of $Y'QY$ is guaranteed when $l'Y'QYl$ is, for every vector l, distributed as a multiple of a chi-square (Corollary 3). Connections between some of the encountered covariance structures and group symmetry covariance models are also briefly discussed.

Usually Wishartness results for matrix quadratic forms $Y'QY$ are established for situations where the variance–covariance matrix Σ_Y of the y_i's admits a Kronecker product representation $\Sigma_Y = B \otimes \Sigma$. There are, however, a number of important instances where Σ_Y cannot be represented in this form, as noted by Anderson, Anderson, and Olkin (1986), Pavur (1987), Mathew (1989), and Wong *et al.* (1991), among others. An example is the multivariate mixed or random effects model, considered by Anderson *et al.* (1986) and by Mathew (1989). In Section 3 we indicate briefly the applicability of our results to such models.

2. RESULTS

We shall first prove a special case of the characterization of Wishartness of $Y'QY$.

THEOREM 1. *Let Y be a random $n \times p$ matrix having the normal distribution $N(0, \Sigma_Y)$. Then $Y'Y \sim W_p(m, I_p)$ if and only if $\Sigma_Y = A \otimes I_p$, where A is a symmetric and idempotent $n \times n$ matrix of rank m.*

Proof. Sufficiency is easily proved and well known; cf., e.g., Proposition 8.5 in Eaton (1983). We shall thus prove only the necessity part. Suppose that $Y'Y \sim W_p(m, I_p)$, and recall from (1.1) that $\Sigma_Y = (\Sigma_{ij})$, where the Σ_{ij}'s are $p \times p$ matrices for $i, j = 1, ..., n$. Let k and l be any $p \times 1$ vectors satisfying $k'l = 0$. Then, since $Y'Y \sim W_p(m, I_p)$, $k'Y'Yk$ and $l'Y'Yl$ are independently distributed, and therefore $\mathrm{Cov}(k'Y'Yk, l'Y'Yl) = 0$. On the other hand,

$$\mathrm{Cov}(k'Y'Yk, l'Y'Yl) = \mathrm{Cov}\left[\sum_{i=1}^{n} (k'y_i)^2, \sum_{j=1}^{n} (l'y_j)^2\right]$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}[(k'y_i)^2, (l'y_j)^2].$$

Now upon rewriting $k'y_i$ as a linear functional of $y = (y_1', ..., y_n')'$, it is seen that $(k'y_i)^2 = y'K_i y$, where K_i is the $np \times np$ block matrix formed from blocks of size $p \times p$, all of which vanish, except the ith diagonal block which is kk'. Similarly $(l'y_j)^2 = y'L_j y$, with ll' as jth diagonal block of L_j. Further,

$$\mathrm{Cov}(y'K_i y, y'L_j y) = 2\,\mathrm{tr}(K_i \Sigma_Y L_j \Sigma_Y),$$

and since, by a routine computation, $\mathrm{tr}(K_i \Sigma_Y L_j \Sigma_Y) = (k'\Sigma_{ij} l)^2$, it follows that

$$\mathrm{Cov}(k'Y'Yk, l'Y'Yl) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (k'\Sigma_{ij} l)^2.$$

Hence $k'\Sigma_{ij} l = 0$ for all $i, j = 1, ..., n$. But $k'\Sigma_{ij} l = 0$ can hold for all vectors k and l satisfying $k'l = 0$ only if Σ_{ij} is a multiple of I_p; i.e., $\Sigma_{ij} = a_{ij} I_p$ for some scalar a_{ij}. Since this must be true for all $i, j = 1, ..., n$, we obtain $\Sigma_Y = A \otimes I_p$, with $A = (a_{ij})$. The matrix A is clearly symmetric; in fact, it is necessarily nonnegative definite.

It remains to be shown that when $\Sigma_Y = A \otimes I_p$, Wishartness of $Y'Y$ implies the idempotency of A. Let $D = \mathrm{diag}(d_1, ..., d_r)$ be the diagonal matrix of nonzero eigenvalues of A, $r = \mathrm{rank}(A)$. Standard canonical reduction arguments then show that $Y'Y = U'DU$, where $U = (u_1, ..., u_r)'$ is a random $r \times p$ matrix distributed as $N(0, I_r \otimes I_p)$. But, since $Y'Y \sim W_p(m, I_p)$, it follows that

$$l'Y'Yl/l'l = \sum_{i=1}^{r} d_i (l'u_i)^2 / l'l$$

has a chi-square distribution with m degrees of freedom for every $l \neq 0$. From univariate theory we therefore obtain that $d_i = 1$ and $r = m$, i.e., $A^2 = A$ and $\text{rank}(A) = m$. This completes the proof of Theorem 1.

Before stating the more general result on Wishartness of $Y'QY$, we introduce some notation. Let w_i be the ith column vector of Y, $i = 1, ..., p$, and let

$$\Theta_{ij} = \text{Cov}(w_i, w_j), \qquad \lambda_{ij} = \text{tr}(Q\Theta_{ij}), \qquad \Lambda = (\lambda_{ij}) \tag{2.1}$$

for $i, j = 1, ..., p$, where Q is the nonnegative definite matrix occurring in the matrix quadratic form $Y'QY$. Note that $\text{E}(Y'QY) = \Lambda$, hence Λ is nonnegative definite.

THEOREM 2. *Let Y be a random $n \times p$ matrix having the normal distribution $N(0, \Sigma_Y)$ and let Λ be the $p \times p$ matrix defined in* (2.1). *Let $Q = TT'$ be a nonnegative definite $n \times n$ matrix of rank r, where T is an $n \times r$ matrix of rank r. Then $Y'QY$ follows a Wishart distribution if and only if one of the following equivalent conditions holds*:

(i) $(T' \otimes I_p)\,\Sigma_Y(T \otimes I_p) = C \otimes \Lambda$, *where mC is a symmetric and idempotent $r \times r$ matrix with* $m = \text{rank}(C)$,

(ii) $(Q \otimes I_p)\,\Sigma_Y(Q \otimes I_p) = (1/m)\,QAQ \otimes \Lambda$, *where A is a nonnegative definite $n \times n$ matrix such that QA is idempotent and* $m = \text{rank}(QA)$.

When (i), *or the equivalent condition* (ii) *holds*, $Y'QY \sim W_p(m, \Sigma)$, *with* $\Sigma = (1/m)\,\Lambda$.

Remark 1. Before proving Theorem 2, we note the following. The conditions in Theorem 2 are easy to verify once the matrix Λ in (2.1) and the matrix T (for verifying condition (i)) have been obtained. For example, using condition (i), one can readily check if the matrix $(T' \otimes I_p)\,\Sigma_Y(T \otimes I_p)$ admits the representation $C \otimes \Lambda$, for some (symmetric) matrix C. The rank of C, say m, can then be computed and the idempotency of mC can be verified. In other words, in order to verify the Wishartness of $Y'QY$ using our conditions in Theorem 2, it is not necessary to specify in advance the scale matrix Σ associated with the Wishart distribution, or the degrees of freedom, as is required in Proposition 3.1 in Wong and Wang (1993). Both these quantities can be determined, once it is known that Wishartness holds. It should also be noted that, while the proof of Proposition 3.1 in Wong and Wang (1993) relies on more advanced techniques such as analytic continuation of moment generating functions, the proof of Theorem 2 given below uses only elementary properties of the Wishart distribution.

Proof of Theorem 2. Suppose that $Y'QY \sim W_p(m, \Sigma)$. Then $\mathrm{E}(Y'QY) = m\Sigma$, and since $\mathrm{E}(Y'QY) = \Lambda$, it follows directly that $\Sigma = (1/m)\,\Lambda$. Let $\operatorname{rank}(\Lambda) = s$, let T be as defined in the statement of the theorem and let $P = (P_1 : P_2)$ be a nonsingular $p \times p$ matrix, where P_1 is $p \times s$ and P_2 is $p \times (p-s)$ such that

$$P'\Sigma P = \operatorname{diag}(I_s, 0). \tag{2.2}$$

Since $Y'QY \sim W_p(m, \Sigma)$, it follows from (2.2) that

$$P'Y'TT'YP \sim W_p(m, \operatorname{diag}(I_s, 0)). \tag{2.3}$$

Let

$$U = T'YP = (T'YP_1 : T'YP_2) = (U_1 : U_2). \tag{2.4}$$

Then (2.3) is equivalent to

$$U_1'U_1 \sim W_s(m, I_s), \qquad U_2 = 0 \quad \text{(with probability one)}. \tag{2.5}$$

From the definition of U_1 and U_2 in (2.4), we obtain the following representations for the variance–covariance matrices of $\operatorname{vec}(U_1')$ and $\operatorname{vec}(U_2')$:

$$\begin{aligned} \operatorname{Var}[\operatorname{vec}(U_1')] &= (T \otimes P_1)'\, \Sigma_Y (T \otimes P_1) \\ \operatorname{Var}[\operatorname{vec}(U_2')] &= (T \otimes P_2)'\, \Sigma_Y (T \otimes P_2). \end{aligned} \tag{2.6}$$

Now, (2.5) and (2.6), along with Theorem 1, yield the condition

$$(T \otimes P_1)'\, \Sigma_Y (T \otimes P_1) = B \otimes I_s, \tag{2.7}$$

where B is a symmetric and idempotent $r \times r$ matrix of rank m. Also, since $U_2 = 0$ (with probability one), $\operatorname{Var}[\operatorname{vec}(U_2')]$ vanishes, and hence

$$(T \otimes P_2)'\, \Sigma_Y = 0. \tag{2.8}$$

The equalities (2.7) and (2.8) can thus together be written as

$$(T \otimes P)'\, \Sigma_Y (T \otimes P) = B \otimes \operatorname{diag}(I_s, 0). \tag{2.9}$$

On the other hand, using (2.2) and the fact that $\Sigma = (1/m)\,\Lambda$, (2.9) can equivalently be written as

$$(T \otimes I_p)'\, \Sigma_Y (T \otimes I_p) = (1/m)\, B \otimes \Lambda. \tag{2.10}$$

Taking $C = (1/m)\, B$, we obtain condition (i). We have thus established the necessity of condition (i).

To establish the equivalence of conditions (i) and (ii), note that $B = T'AT$, for some nonnegative definite $n \times n$ matrix A. (For example, one can take $A = T(T'T)^{-1} B(T'T)^{-1} T'$.) Since $Q = TT'$, (2.10) can equivalently be expressed as

$$(Q \otimes I_p)\, \Sigma_Y (Q \otimes I_p) = (1/m)\, QAQ \otimes A. \tag{2.11}$$

The idempotency of $B = T'AT$ is equivalent to that of QA. Furthermore, $\operatorname{rank}(B) = \operatorname{rank}(T'AT) = \operatorname{rank}(QA)$. We have thus established the equivalence of the conditions (i) and (ii) and have also established their necessity for the Wishartness of $Y'QY$. Since all the arguments in the proof can be reversed, the sufficiency part follows as well. The sufficiency can also be deduced by using Proposition 8.5 in Eaton (1983). This completes the proof of Theorem 2.

Nonnegative definite solutions for Σ_Y can be obtained by solving either one of the matrix equations in Theorem 2, using Lemma 2.1 in Khatri and Mitra (1976). This yields the following result.

Corollary 1. *Let Y be a random $n \times p$ matrix having the normal distribution $N(0, \Sigma_Y)$, let Q be a nonnegative definite $n \times n$ matrix, and let P_Q denote the orthogonal projection matrix projecting onto the range of Q. Then $Y'QY$ follows a Wishart distribution if and only if Σ_Y admits the representation*

$$\Sigma_Y = A \otimes \Sigma + (I_{np} - P_Q \otimes I_p)\, N(I_{np} - P_Q \otimes I_p), \tag{2.12}$$

where A is a nonnegative definite $n \times n$ matrix such that QA is idempotent, Σ is a nonnegative definite $p \times p$ matrix, and N is a nonnegative definite $np \times np$ matrix. In particular, when Q is positive definite, $Y'QY$ follows a Wishart distribution if and only if

$$\Sigma_Y = A \otimes \Sigma, \tag{2.13}$$

where A and Σ are as in (2.12). *Also, when* (2.12) *or* (2.13) *holds, $Y'QY \sim W_p(m, \Sigma)$, with $m = \operatorname{rank}(A)$.*

Remark 2. Corollary 1 provides an explicit and complete characterization of the class of possible covariance structures Σ_Y under which the distribution of $Y'QY$ is Wishart. Thus, e.g., for positive definite weight matrix Q, the only permissible covariance structures are those with variance and covariance blocks of Σ_Y given by

$$\Sigma_Y = (a_{ij} \Sigma), \tag{2.14}$$

where the scalar multiples a_{ij} are the elements of the matrix A. In particular, when $Q = I_n$, the matrix A is symmetric and idempotent, and its element a_{ij} are constrained by a number of well-known bounds and relationships, most of which follow at once from $a_{ii} = \sum_{j=1}^{n} a_{ij}^2$ $(i = 1, ..., n)$. (With suitable modifications, this is true also for a general positive definite Q, but the corresponding bounds and relationships involve the elements of Q as well, and are therefore less transparent.) In view of (2.14), the variance and covariance blocks of Σ_Y exhibit necessarily a strong structural uniformity, the precise form of which is inherited from the bounds on, and the relationships between, the elements a_{ij} of the matrix A. It should also be noted that, except for the standard case when the y_i's are i.i.d. $N(0, I_p)$, the matrix Σ_Y will invariably be singular, the singularity arising from the structure of Σ_Y exhibited in (2.14).

The proof of Theorem 1 uses the following facts about the $W_p(m, I_p)$ distribution: if $S \sim W_p(m, I_p)$, then

(i) $k'Sk$ and $l'Sl$ are independently distributed provided $k'l = 0$, and

(ii) $l'Sl/l'l$ follows a chi-square distribution with m degrees of freedom.

In particular, the idempotency of A in Theorem 1 was seen to follow from property (ii).

It is known that property (ii) alone does not, in general, characterize the Wishart distribution; see Mitra (1969). However, if the y_i's, that is, the columns of the random matrix Y', are independently distributed with a common covariance matrix Σ, then it is true that

$$Y'QY \sim W_p(m, \Sigma) \Leftrightarrow l'Y'QYl/l'\Sigma l \sim \chi_m^2 \tag{2.15}$$

for every l satisfying $l'\Sigma l \neq 0$; see result 8b.2(ii) in Rao (1973). Theorems 1 and 2 deal with random matrices of the type $Y'QY$, but with a general covariance structure Σ_Y. We shall now show that (2.15) fails, in general, under this more general covariance setup. We shall, in fact, obtain a necessary and sufficient condition for $l'Y'QYl$ to be distributed as a multiple of a chi-square random variable for every vector l. The condition is weaker than the conditions in Theorem 2 and is proved below when Σ_Y is positive definite.

We note that when Σ_Y is positive definite, so is $(T' \otimes I_p)\, \Sigma_Y (T \otimes I_p)$, where T is as defined in Theorem 2. Consequently, C in Theorem 2(i) is nonsingular. This will force $rC = I_r$, since $m = \text{rank}(C) = r$ and rC is idempotent. Thus, in the special case when Σ_Y is positive definite, the condition in Theorem 2(i) simplifies to $(T' \otimes I_p)\, \Sigma_Y (T \otimes I_p) = I_r \otimes (1/r)\, \Delta$; i.e., the matrix $\Delta = (T' \otimes I_p)\, \Sigma_Y (T \otimes I_p)$ is block diagonal with equal blocks $(= \Sigma)$.

This may be contrasted with the structure for Δ, emerging from Theorem 3 below.

THEOREM 3. *Let Y be a random $n \times p$ matrix having the normal distribution $N(0, \Sigma_Y)$ and assume that Σ_Y is positive definite. Let $Q = TT'$ be a nonnegative definite $n \times n$ matrix of rank r, where T is an $n \times r$ matrix of rank r. Define $\Delta = (T' \otimes I_p)\, \Sigma_Y (T \otimes I_p)$ and partition it as $\Delta = (\Delta_{ij})$, where the Δ_{ij}'s are $p \times p$ matrices, $i, j = 1, ..., r$. Then $l'Y'QYl$ is distributed as a multiple of a chi-square distribution for every vector l if and only if $\Delta_{11} = \cdots = \Delta_{rr}$ and the Δ_{ij}'s are skew-symmetric matrices for $i \neq j$, $i, j = 1, ..., r$.*

Proof. Let $U = T'Y$. Then U is an $r \times p$ matrix and $\mathrm{Var}[\mathrm{vec}(U')] = \Delta$, where Δ is as defined in the theorem. Also, $l'Y'QYl = l'U'Ul$ and

$$\mathrm{Var}(Ul) = (I_r \otimes l')\, \Delta (I_r \otimes l) = (l' \Delta_{ij} l).$$

Hence, $Ul \sim N(0, \Delta_l)$, where Δ_l denotes the matrix with $l'\Delta_{ij} l$ as (i, j)th element. Thus $l'U'Ul$ is distributed as a multiple of a chi-square for every l if and only if Δ_l is a multiple of a symmetric and idempotent matrix for every l. Note that since Σ_Y is positive definite, by assumption, so is Δ, and therefore Δ_l is positive definite for every $l \neq 0$. Hence Δ_l is, for every l, a multiple of a symmetric and idempotent matrix if and only if, for every l, it is a multiple of the identity matrix. Thus

$$l'\Delta_{11} l = \cdots = l'\Delta_{rr} l, \qquad l'\Delta_{ij} l = 0 \qquad \text{for} \quad i \neq j$$

$(i, j = 1, ..., r)$. But these conditions can hold for all l if and only if

$$\Delta_{11} = \cdots = \Delta_{rr}, \qquad \Delta_{ij} = -\Delta'_{ij}, \qquad i \neq j;$$

i.e., the Δ_{ij}'s are skew-symmetric for $i \neq j$ $(i, j = 1, ..., r)$. This completes the proof of Theorem 3.

Remark 3. Since the block matrix Σ_Y will typically exhibit structural singularities (cf. Remark 2), it would be of interest to relax the assumption that Σ_Y be positive definite. However, even in the simplest cases, this leads to a somewhat intractable linear-algebraic problem for which no useful necessary and sufficient condition appears to be available. On the other hand, it should be noted that the condition in Theorem 3 is, of course, still sufficient when Σ_Y is only assumed to be nonnegative definite.

If in Theorem 3 we choose $Q = I_n$, then Δ (as defined there) reduces to the variance–covariance matrix Σ_Y, yielding the following result.

COROLLARY 2. *Let $Y = (y_1, ..., y_n)'$ be a random $n \times p$ matrix having the normal distribution $N(0, \Sigma_Y)$ and assume that Σ_Y is positive definite. Then*

$l'Y'Yl$ is distributed as a multiple of a chi-square distribution for every vector l if and only if Σ_Y is of the form

$$\Sigma_Y = \begin{pmatrix} \Sigma & -\Sigma_{21} & \cdots & -\Sigma_{n1} \\ \Sigma_{21} & \Sigma & \cdots & -\Sigma_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \cdots & \Sigma \end{pmatrix}, \tag{2.16}$$

that is, if and only if the marginal distribution of every pair $(y_i', y_j')'$, $i \neq j$, $i, j = 1, ..., n$, is a p-variate complex normal distribution.

Mitra (1969) has given a counterexample which shows that, for a random $p \times p$ matrix S, Wishartness of S does not follow if $l'Sl$ is distributed as a multiple of a chi-square for every vector l. Indeed, Mitra (1969) exhibited a random $p \times p$ matrix XS, with X independent of S and univariate beta distributed and S Wishart distributed and showed that chi-squaredness holds for XS. However, assuming Wishartness for XS was shown to be in contradiction with the Wishartness for S. To the author's knowledge, this counterexample is the only one available in the literature and is frequently reproduced in textbooks on multivariate analysis (see, e.g., Mardia, Kent, and Bibby (1979, p. 89) and Seber (1984, pp. 20–21).

Using Corollary 2 together with Theorem 1 we shall derive below another counterexample. The counterexample is implicit in Theorem 3 and Corollary 2. Our counterexample complements that of Mitra (1969) in two ways. First, the counterexample will be by explicit construction. Second, it will show that Wishartness does not follow even when S is a matrix quadratic form involving a normally distributed random matrix.

EXAMPLE 1. Suppose that y_1 and y_2 are bivariate random vectors such that $Y = (y_1, y_2)'$ is a 2×2 random matrix having a normal distribution with mean zero and the following covariance structure:

$$\operatorname{Var}(y_1) = \operatorname{Var}(y_2) = I_2, \qquad \operatorname{Cov}(y_1, y_2) = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}. \tag{2.17}$$

Then

$$\Sigma_Y = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{pmatrix}$$

is a positive definite matrix. Since the off-diagonal block in (2.17) is skew-symmetric, it follows from Corollary 2 that $l'Y'Yl$ is distributed as a multiple of a chi-square random variable for every vector l. In fact, $l'Y'Yl/l'l$ has a chi-square distribution with two degrees of freedom for every $l \neq 0$. This is also obvious by noting that $Yl/\sqrt{l'l} \sim N(0, I_2)$ for every $l \neq 0$. However, $Y'Y$ does not have a Wishart distribution, as is clear from Theorem 1. One can immediately see this by noting that if $Y'Y$ does have a Wishart distribution, then it must be $W_2(2, I_2)$ (since $\mathrm{E}(Y'Y) = 2I_2$). This, in particular, implies that the two diagonal elements of $Y'Y$ are independently distributed. It is readily verified that this is not the case. In fact, direct computations show that the covariance between the two diagonal elements of $Y'Y$ is nonzero and is equal to $\frac{1}{2}$. (That $Y'Y$ cannot possibly have a Wishart distribution is also seen directly from the fact that the constructed Σ_Y does not admit a representation of the form $A \otimes I_2$.)

The following corollary exhibits a situation under which Wishartness of $Y'QY$ is guaranteed when $l'Y'QYl$ is distributed as a multiple of a chi-square random variable for every vector l.

COROLLARY 3. *Let $Y = (y_1, ..., y_n)'$ be a random $n \times p$ matrix having the normal distribution $N(0, \Sigma_Y)$ and assume that Σ_Y is positive definite. Assume further that $\mathrm{Cov}(y_i, y_j)$ is a symmetric matrix for all $i, j = 1, ..., n$. Then $Y'QY$ has a Wishart distribution if and only if $l'Y'QYl$ is distributed as a multiple of a chi-square random variable for every vector l.*

Proof. When $\mathrm{Cov}(y_i, y_j)$ is symmetric for all i and j, it is readily verified that the off-diagonal blocks Δ_{ij} of the matrix Δ, defined in Theorem 3, are all symmetric. Hence, if $l'Y'QYl$ is distributed as a multiple of a chi-square random variable for every vector l, Theorem 3 along with the symmetry of the Δ_{ij}'s yields $\Delta_{11} = \cdots = \Delta_{rr} = \Delta_0$, say, and $\Delta_{ij} = 0$ for all $i \neq j$. Hence $\Delta = I_r \otimes \Delta_0$, and since $\mathrm{E}(Y'QY) = r\Delta_0$, it follows directly from Theorem 2 that $Y'QY$ follows a Wishart distribution.

Although we shall not pursue this in detail here, we note that there are interesting connections between several of the above covariance structures and group symmetry covariance models. A group symmetry covariance model is a class of covariance matrices Σ, each of which remains invariant in the usual sense $G\Sigma G' = \Sigma$ for all G in $\mathcal{G}$, a finite group of orthogonal matrices; for further details; see, e.g., Eaton (1983, Chap. 9) or Perlman (1987) and the references therein. For example, the structure (2.16), appearing in Corollary 2, amounts to requiring symmetry of the covariance matrix of the marginal distribution of every pair $(y_i', y_j')'$, $i \neq j$, with respect to the four-element group $\mathcal{G} = \{\pm I_{2p}, \pm K_{2p}\}$, where

$$K_{2p} = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}.$$

For obvious reasons the covariance matrix of $(y_i', y_j')'$ is said to exhibit a complex covariance structure; for further details; see, e.g., Eaton (1983, Chap. 9).

On the other hand, the covariance structure assumed in Corollary 3 is general enough to contain several important group symmetry models. An example of such a model is the dihedral block symmetry model, which combines ordinary circular block symmetry

$$\mathrm{Cov}(y_1', y_2', ..., y_n')' = \mathrm{Cov}(y_2', ..., y_n', y_1')' = \cdots = \mathrm{Cov}(y_n', y_1', ..., y_{n-1}')'$$

with the requirement that $\mathrm{Cov}(y_i, y_j) = \mathrm{Cov}(y_j, y_i)$ for all $i, j = 1, ..., n$. E.g., in the case $n=4$, a covariance matrix Σ_Y satisfying dihedral block symmetry thus exhibits the structure

$$\Sigma_Y = \begin{pmatrix} \Sigma_0 & \Sigma_1 & \Sigma_2 & \Sigma_1 \\ \Sigma_1 & \Sigma_0 & \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_1 & \Sigma_0 & \Sigma_1 \\ \Sigma_1 & \Sigma_2 & \Sigma_1 & \Sigma_0 \end{pmatrix},$$

a special case of the covariance structure assumed in Corollary 3 with $n=4$. For further details, including identification of the corresponding group; see Perlman (1987) and the references therein.

Another example is the complete block symmetry model, which prescribes that $\mathrm{Cov}(y_i) = \mathrm{Cov}(y_j)$ and $\mathrm{Cov}(y_i, y_j) = \mathrm{Cov}(y_k, y_l)$ for all $i \neq j$ and $k \neq l$; and it is the block version of the intraclass correlation model. In this case Σ_Y exhibits the structure

$$\Sigma_Y = \begin{pmatrix} \Sigma_0 & \Sigma_1 & \cdots & \Sigma_1 \\ \Sigma_1 & \Sigma_0 & \cdots & \Sigma_1 \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_1 & \Sigma_1 & \cdots & \Sigma_0 \end{pmatrix}, \tag{2.18}$$

which again is a special case of the covariance structure assumed in Corollary 3.

Remark 4. Throughout we have only considered the central case, i.e., the case when mean of Y is zero. In the noncentral case with $Y \sim N(M_Y, \Sigma_Y)$, $M_Y \neq 0$, the necessary and sufficient conditions for $Y'QY$ to follow a noncentral Wishart distribution include a condition on the mean matrix M_Y as well. In fact, the conditions derived in Wong *et al.* (1991) and Wong and Wang (1993, 1995) are for the noncentral case; e.g., Proposition 3.1 in Wong and Wang (1993). The analogue of Theorem 2 in the noncentral case is as follows.

Let $Y \sim N(M_Y, \Sigma_Y)$. Then $Y'QY$ has a noncentral Wishart distribution if and only if (i) any one of the equivalent conditions in Theorem 2 hold, and (ii) $M'_YQM_Y = M'_YQAQM_Y$, where A is the matrix given in Theorem 2(ii) (cf. condition (a3) in Proposition 3.1 in Wong and Wang (1993)). When these two conditions hold, the noncentrality matrix is given by M'_YQM_Y. In the setup of Theorem 1 with $\mathrm{E}(Y) = M_Y \neq 0$, the proof of Theorem 1 can be easily modified to arrive at the necessity of the conditions (i) and (ii) given above. One has to use the conditions under which a univariate quadratic form has a noncentral chisquare distribution; see, e.g., Theorem 9.2.1 in Rao and Mitra (1971). Once the result is thus established in the setup of Theorem 1, it also follows for Theorem 2 (with $\mathrm{E}(Y) = M_Y \neq 0$), since Theorem 2 is proved by reducing it to the setup of Theorem 1.

Conditions for the independence of several matrix quadratic forms and related multivariate versions of Cochran's theorem can be found, e.g., in Wong *et al.* (1991) and Wong and Wang (1993); see also references in these articles.

3. APPLICATIONS TO MULTIVARIATE COMPONENTS OF VARIANCE MODELS

The multivariate components of the variance model are given by

$$Y = XB + \sum_{i=1}^{k-1} A_i W_i + E, \tag{3.1}$$

where $Y = (y_1, ..., y_n)'$ is the $n \times p$ matrix of observations; X is a known $n \times s$ matrix; A_i is a known $n \times s_i$ matrix; B is an $s \times p$ matrix of unknown parameters; W_i is a random $s_i \times p$ matrix whose rows are independently and normally distributed with mean zero and covariance matrix Σ_i, $i = 1, ..., k-1$; and E is a random $n \times p$ matrix whose rows are independently and normally distributed with mean zero and covariance matrix Σ_k, assumed to be positive definite. It is also assumed that the W_i's and E are all independently distributed. Thus, Y is a normally distributed random matrix with

$$\mathrm{E}(Y) = XB, \qquad \mathrm{Var}[\mathrm{vec}(Y')] = \sum_{i=1}^{k-1} (A_i A'_i \otimes \Sigma_i) + I_n \otimes \Sigma_k. \tag{3.2}$$

Since the matrices $A_i A'_i$ are symmetric, it is clear from (3.2) that $\mathrm{Cov}(y_i, y_j)$ is symmetric for all $i, j = 1, ..., n$, and therefore the assumption of Corollary 3 is satisfied. Consequently, in order to verify the Wishartness of $Y'QY$ (with Q a nonnegative definite $n \times n$ matrix) it is enough to verify if, for every vector l, $l'Y'QYl$ is distributed as a multiple of a chi-square. In the present situation, Yl is a random $n \times 1$ vector following the univariate components of variance model

$$Yl = XBl + \sum_{i=1}^{k-1} A_i W_i l + El, \tag{3.3}$$

with

$$\mathrm{E}(Yl) = XBl, \qquad \mathrm{Var}(Yl) = \sum_{i=1}^{k-1} \sigma_i^2(l)\, A_i A_i' + \sigma_k^2(l)\, I_n, \tag{3.4}$$

the univariate components of variance being $\sigma_i^2(l) = l'\Sigma_i l$, $i = 1, ..., k$. Therefore, in order to establish Wishartness for matrix quadratic forms $Y'QY$ in the multivariate components of variance model (3.1), it suffices to establish chi-squaredness (up to a multiple) for the ordinary quadratic form $l'Y'QYl$ in the univariate components of variance model (3.3). This observation applies to the independence of matrix quadratic forms as well; see the condition for independence in Wong *et al.* (1991) and Wong and Wang (1993).

A special case of (3.1), exhibiting even further symmetry in the covariance matrix, is the multivariate linear model with exchangeable covariance structure. In this case the covariance matrix Σ_Y of vec(Y') is of the form

$$\Sigma_Y = 1_n 1_n' \otimes \Sigma_1 + I_n \otimes (\Sigma_0 - \Sigma_1);$$

i.e., admits the structure (2.18); for further details see Mathew (1989) and the references therein.

More generally, we note that the multivariate mixed and random effects models, with balanced as well as unbalanced data, are all special cases of (3.1). Hence, Wishartness and independence of matrix quadratic forms in such models will follow from the corresponding chi-squaredness and independence results in the univariate models. The model (3.1) has been studied by Anderson *et al.* (1986) and by Mathew (1989).

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